Thermal flow in the gravitational $O(n)$ model

Ivan K. Kostov[1][2]

Service de Physique Théorique, CNRS – URA 2306
C.E.A. - Saclay,
F-91191 Gif-sur-Yvette, France

Abstract

We study the massless flow from the critical point (dilute loops) to the low-temperature phase (dense loops) of the $O(n)$ loop gas model when the model is coupled to 2D gravity. The flow is generated by the gravitationally dressed thermal operator $\Phi_{1,3}$ coupled to the renormalized loop tension $\lambda \sim T - T_c$. We find that the susceptibility as a function of the thermal coupling λ and the cosmological constant μ satisfies a simple transcendental equation.

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[1] Ivan.Kostov@cea.fr
[2] Associate member of INRNE – BAS, Sofia, Bulgaria

1 Introduction and summary

It is well known that the critical phenomena on flat and fluctuating lattices are deeply related. For each critical point described by a "matter" CFT, the "coupling to gravity" consists in adding a Liouville and ghost sectors and dressing the scaling operators by exponents of the Liouville field. In such simplest theories of 2D gravity are quite well understood nowadays due to the progress towards the exact solution of Liouville theory achieved in the last decade.

On the other hand, almost nothing is known about theories of 2D gravity in which the matter field has massless excitations but is not conformal invariant. A typical example of such a theory is the $O(n)$ loop model on a honeycomb lattice [1]. This model has two non-trivial critical points, the dense and the dilute phases of the loop gas, described by two different CFT's. The massless flow relating these two CFT's is generated by the thermal operator $\Phi_{1,3}$. Other theories with massless flows have been studied in [2–4].

The thermal flow in the gravitational $O(n)$ model is expected to be described by Liouville and matter CFT's coupled through the operator gravitationally dressed thermal operator $\Phi_{1,3}$. It is not known at present how to solve such a theory. On the other hand, the corresponding microscopic theory, the $O(n)$ model on random planar graphs, can be solved exactly using its dual formulation as a matrix model [5–8]. The exact solution in the case of general potential has been formulated in [9, 10].

The aim of this paper is to work out, using the matrix model formulation, explicit expressions for the partition functions of the gravitational $O(n)$ model on the disk and on the sphere along the thermal flow. In this section we give a brief introduction in the $O(n)$ model and present our results.

The $O(n)$ model can be defined on any trivalent graph $\mathcal{G}$. The local fluctuating variable associated with the sites $r \in \mathcal{G}$ is an $O(n)$ spin with components $S_1(r), \ldots S_n(r)$, normalised so that $\mathrm{Tr}\, S_a(r)S_b(r') = \delta_{ab}\delta_{rr'}$. The partition function on the graph $\mathcal{G}$ is defined as

$$Z_{O(n)}(\mathcal{G}; T) = \mathrm{Tr} \prod_{<rr'>} \left(1 + \tfrac{1}{T}\sum_a S_a(r)S_a(r')\right) \tag{1.1}$$

where T is the temperature and the product runs over all links $<rr'>$ of the graph $\mathcal{G}$. Expanding the trace as a sum of monomials, the partition function can be written as a sum over all configurations of self-avoiding, mutually avoiding loops that can be drawn on $\mathcal{G}$,

$$Z_{O(n)}(\mathcal{G}, T) = \sum_{\text{loops}} T^{-\mathcal{L}_{\text{tot}}}\, n^{\mathcal{N}_{\text{loops}}}, \tag{1.2}$$

as shown in Fig.1. Here $\mathcal{L}_{\text{tot}}$ is the total length of the loops, equal to the number of occupied lattice edges and $\mathcal{N}_{\text{loops}}$ is the number of loops. Unlike the original formulation, the loop gas representation (1.2) makes sense also for non-integer n and has a continuous transition in two dimensions for $|n| \leq 2$. In this interval the number of flavors can be parametrized as
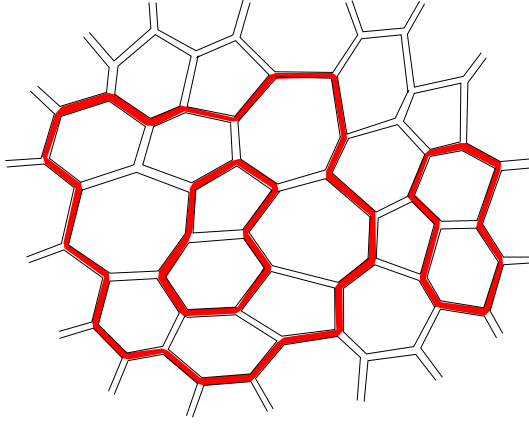
Fig. 1: Loops on a trivalent fat planar graph

$$n = 2\cos\pi\nu, \quad 0 < \nu < 1. \tag{1.3}$$

The phase diagram of the loop gas on the infinite regular trivalent graph, the honeycomb lattice, was first established in [1]. At the critical temperature $T_c = 2\cos\frac{\pi}{4}\nu$ the loop gas model is solvable and is described by a CFT with central charge

$$c_{\text{critical}} = 1 - 6\frac{\nu^2}{1+\nu}. \tag{1.4}$$

For $T > T_c$ the theory has a mass gap. The low-temperature, or "dense", phase $T < T_c$ is a flow to an attractive fixed point [18] at $T_c^{\text{dense}} = 2\sin\frac{\pi}{4}\nu$, where the theory is again solvable and is described by a CFT with smaller central charge

$$c_{\text{dense}} = 1 - 6\frac{\nu^2}{1-\nu}. \tag{1.5}$$

The scaling behaviour of the model in the vicinity of a critical point is described by an action of the form [4]

$$\mathcal{A} = \mathcal{A}_{\text{critical}} + \delta T \int \Phi_{1,3} \tag{1.6}$$

where $\delta T = T - T_c$ and $\Phi_{1,3}$ is the thermal operator with conformal dimensions

$$\Delta_{1,3} = \bar{\Delta}_{1,3} = \frac{1-\nu}{1+\nu}. \tag{1.7}$$

Microscopically the thermal operator $\Phi_{1,3}$ counts the total length of the loops $\mathcal{L}_{\text{tot}}$. Added to the action, it generates a mass of the loops. For $\delta T > 0$ the deformation (1.6) describes, going from short to the long distance scales, the flow to a massive theory with mass gap

$$m \sim \delta T^{1/(2-2\Delta_{1,3})} = \delta T^{\frac{1+\nu}{4\nu}}. \tag{1.8}$$

When $\delta T < 0$ the deformation (1.6) describes a massless flow between two different CFT with central charges (1.4) and (1.5). In the CFT for the dense phase, describing the IR limit, the flow to the attractive critical point is generated by another, irrelevant, operator, $\Phi_{3,1}$.

The $O(n)$ model on a fluctuating lattice [5] can be formulated as a statistical ensemble of trivalent planar graphs covered by self-avoiding and mutually avoiding loops. The critical thermodynamics is controlled by the temperature T and one extra parameter, the bare cosmological

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constant κ coupled to the size of the graph. For example, the partition function on the sphere is defined as

$$\mathcal{F}(\kappa,T) = \sum_{\mathcal{G}} \kappa^{\mathcal{A}} \, Z_{O(n)}(\mathcal{G},T) \tag{1.9}$$

where the summation is taken over all connected fat graphs $\mathcal{G}$ with the topology of a sphere and $\mathcal{A}$ is the area (the number of vertices) of the graph.

As in the case of a flat lattice, the model has three critical points. Each critical point is characterized with the "string susceptibility" exponent γ_{str} related to the matter central charge by $c_{\mathrm{matter}} = 1 - 6\gamma_{\mathrm{str}}^2/(1-\gamma_{\mathrm{str}})$. [11]. Qualitatively, at the critical point $\kappa = \kappa^*, T = T^*$ both the area of the graph and the length of the loops diverge. The flow to the massive, high-temperature, phase is along the critical line $\kappa = \kappa_I(T)$ where the area of the graph diverges, while the loops remain finite. The flow to the dense, low-temperature, phase is along another line, $\kappa = \kappa_{II}(T)$, where the area of the graph diverges because of the diverging length of the densely packed loops. Thus the continuum limit is described by the vicinity of the critical line $\kappa = \kappa_c(T)$, which consists of two branches meeting at the critical point:

$$\kappa = \kappa_c(T) = \begin{cases} \kappa_I(T), & T > T^*; \\ \kappa_{II}(T), & T < T^*. \end{cases}$$

The exponent γ_{str} is given in the three phases by

$$\gamma_{\mathrm{str}} = \begin{cases} -\frac{1}{2} & \text{high-temperature phase (massive loops);} \\ -\nu & \text{critical point (dilute loops);} \\ -\frac{\nu}{1-\nu} & \text{low-temperature phase (dense loops).} \end{cases} \tag{1.10}$$

In the continuum limit the loop gas model on a fluctuating lattice is described in terms of the renormalized coupling constants

$$\mu \sim \kappa_c(T) - \kappa^2, \qquad \lambda \sim T^* - T. \tag{1.11}$$

Denote by $\Phi(\mu, \lambda, \mu_B)$ the partition function on the disk with boundary cosmological constant μ_B. It is given by the Laplace transform of the disk partition function $\tilde{\Phi}(\mu, \lambda, \ell)$ with fixed boundary length ℓ:

$$\Phi(\mu, \lambda, \mu_B) = \int_0^{\infty} d\ell \, e^{-\mu_B \ell} \, \tilde{\Phi}(\mu, \lambda, \ell). \tag{1.12}$$

An important characteristics of the model is the boundary entropy $-M$, defined as the exponent in the exponential decay of the disk partition function when the length tends to infinity:

$$M = -\lim_{\ell \to \infty} \frac{\log \tilde{\Phi}(\mu, \lambda, \ell)}{\ell}. \tag{1.13}$$

As each loop can be considered as two boundaries glued together, the effective loop tension is equal to twise the boundary entropy.

Summary of the results:

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We have found that the susceptibility $\chi \sim -\partial_\mu^2 \mathcal{F}$ is related to the boundary entropy as

$$\chi = M^{2\nu}. \tag{1.14}$$

The function $M = M(\lambda, \mu)$ is determined by comparing the expressions for the derivative $\partial_\mu \Phi|_{\mu_B}$ and $\partial_{\mu_B} \Phi_\mu$, which we obtained by solving the saddle point equations for the $O(n)$ matrix model. The derivatives of the disk partition function are given in parametric form by

$$\mu_B = M \cosh \tau,$$

$$\partial_{\mu_B} \Phi|_\mu = -2M^{1+\nu} \cosh(1+\nu)\tau$$

$$+2\lambda\, M^{1-\nu} \cosh(1-\nu)\tau,$$

$$\partial_\mu \Phi|_{\mu_B} = \nu^{-1} M^\nu \cosh \nu\tau. \tag{1.15}$$

(The normalization of the coupling constants is chosen for our convenience.) The compatibility of these two expressions implies the following transcendental equation for the boundary entropy:

$$\mu = (1+\nu)\, M^2 + \lambda\, M^{2-2\nu}. \tag{1.16}$$

Equation (1.16) generalizes the previously obtained results for the scaling behavior of the boundary entropy in the dilute and the dense phases, $M \sim \mu^{\frac{1}{2}}$ in the dilute phase and $M \sim \lambda^{\frac{1}{2-2\nu}}$ in the dense phase [12].

As the susceptibility is related to the boundary entropy by (1.14), it satisfies the transcendental equation

$$\mu = (1+\nu)\chi^{\frac{1}{\nu}} + \lambda\chi^{\frac{1-\nu}{\nu}}, \tag{1.17}$$

which coincides, after a redefinition of the variables, with the eqation found in [13] for the susceptibility of the gravitational sine-Gordon model.

The dimension of the coupling $\lambda \sim \mu^\nu$ matches the gravitational dimension $\delta_{1,3} = 1 - \nu$, obtained from $\Delta_{1,3}$ by the KPZ scaling relation $\Delta = \frac{\delta(\delta+\nu)}{1+\nu}$ [11]. This is consistent with the conjecture that for finite λ/μ^ν the theory is described by a perturbation of the critical point $\lambda = 0$ by the Liouville-dressed thermal operator $\Phi_{1,3}$. An analysis of eq. (1.17) in the framework of Liouville gravity was performed by Al. Zamolodchikov [14]. After integration one can reconstruct the expansion of the partition function $\mathcal{F}$ in λ, which generates the n-point correlation functions of the thermal operator. The result for $n \leq 4$ matches with the calculations recently published by A. Belavin and Al. Zamolodchikov [15] as well as with the formula conjectured in [16] on the basis of the ground ring identities. We leave the analysis of the case of a strong perturbation from the perspective of the worldsheet CFT to a future publication [17].

2 Solution of the $O(n)$ matrix model in the planar limit

2.1 The $O(n)$ matrix model

The partition function (1.9) of the $O(n)$ model on random triangulations is generated as the perturbative (t'Hooft) expansion of the free energy of a model of $n+1$ hermitian $N \times N$ matrix

variables X and $\vec{Y} = \{Y_1, ..., Y_n\}$ [5]:

$$\epsilon^{\mathcal{F}} \sim \int dX \prod_{a=1}^{n} dY_a \ e^{N\kappa^{-2}\left(-\frac{1}{2}\text{tr } X^2 - \frac{1}{2}\sum_{a=1}^{n} Y_a^2 + \frac{1}{3}X^3 + \frac{1}{2T}\sum_{a=1}^{n} XY_a^2\right)}. \tag{2.18}$$

We are interested in the planar limit $N \to \infty$, where only genus zero planar Feynman graphs survive. Each such Feynman grah realizes a loop configuration in the sum (1.9).

Integrating out the Y-variables and shifting $X \to X + \frac{1}{2}T$ one obtain one-matrix integral of the form

$$\epsilon^{\mathcal{F}} = \int dX \ e^{-\beta\,\text{tr } V(X)} \ |\text{Det } (1 \otimes X + X \otimes 1)|^{-n/2} \tag{2.19}$$

where $\beta = N/\kappa^2$ and the coefficients of the cubic potential

$$V(X) = t_1 X + t_2 X^2 + t_3 X^3 \tag{2.20}$$

are expressed in terms of the temperature T as

$$t_1 = \frac{T(2-T)}{4}, \quad t_2 = \frac{1-T}{2}, \quad t_3 = -\frac{1}{3}. \tag{2.21}$$

It will be convenient to take the large N limit by sending β to infinity keeping the ratio $\kappa^2 = \beta/N$ finite.

After diagonalization $gXg^{-1} = \{x_1, ..., x_N\}, \quad g \in SU(N)$, the matrix integral (2.19) can be reformulated as a two-dimensional Coulomb gas of N charges constrained at the real axis:

$$e^{\mathcal{F}} \sim \int_{\mathbb{R}} \prod_{i=1}^{N} dx_i \ e^{-\beta V(x_i)} \prod_{i<j}^{N}(x_i - x_j)^2 \prod_{i,j=1}^{N} (x_i + x_j)^{-\frac{n}{2}}. \tag{2.22}$$

In the limit $N \to \infty$ the integral is saturated by the saddle point described by the classical spectral density $\rho(x)$ supported by the interval $[-a, -b]$ on the negative semi-axis, where $0 < b < a$. We will normalize the spectral density as

$$\int_{-a}^{-b} dx \, \rho(x) = \frac{N}{\beta} = \kappa^2. \tag{2.23}$$

The spectral density is determined from the saddle point equation

$$V'(x) = 2\mathcal{P} \int_{-a}^{-b} dy \frac{\rho(y)}{x - y} - n \int_{-a}^{-b} dy \frac{\rho(y)}{x + y}, \qquad x \in [-a, -b]. \tag{2.24}$$

The edges of the eigenvalue distribution are functions of the external potential, $i.e.$ of the temperature T and the cosmological constant κ.

The disk partition function with boundary length ℓ is defined as

$$\tilde{\Phi}(\ell) = \frac{1}{\beta} \text{tr } e^{\ell X} = \frac{1}{\beta} \int_{-a}^{-b} e^{\lambda \ell} \rho(\lambda) d\ell. \tag{2.25}$$

The position of the right endpoint, $-b$, determines the large ℓ asymptotics of the disk partition function, $\Phi(\ell) \sim e^{-b\ell}$, and has the statistical meaning of boundary entropy for unit length. The boundary entropy is negative because shifting the matrix variable X we effectively performed a subtraction.

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Knowing the position of the edge $b = b(T, \kappa)$, the equations of the two branches of the critical line mentioned in the Introduction are obtained as follows [5]. The branch $\kappa = \kappa_{II}(T)$ is determined by the condition that the right endpoint reaches the origin:

$$b = 0 \quad \Rightarrow \quad \kappa_c = \kappa_{II}(T). \tag{2.26}$$

When $b > 0$ the loops are not critical, the critical behavior is that of pure gravity ($c_{\text{matter}} = 0$) and the position $\partial_\kappa b \sim \partial_\kappa^2 \mathcal{F}$ must behave as $(\kappa_c - \kappa)^{-1/2}$. The branch $\kappa_c = \kappa_I(T)$ is determined by the condition that the position of the endpoint b develops a singularity in κ:

$$\frac{\partial b}{\partial \kappa}\bigg|_T \to \infty \quad \Rightarrow \quad \kappa_c = \kappa_I(T). \tag{2.27}$$

Finally, the critical point $\{T, \kappa\} = \{T^*, \kappa^*\}$ is the common endpoint of the two critical lines: $k^* = \kappa_I(T^*) = \kappa_{II}(T^*)$.

2.2 Functional equation for the loop field

The saddle point equation can be solved directly, see *e.g.* [6] for the solution in the special case $b = 0$. It is however more advantageous to reformulate the problem in terms of the collective field, or *loop operator*,

$$\Phi(x) = -\tfrac{1}{\beta} \sum_{i=1}^{N} \log(x - x_i), \tag{2.28}$$

which is the Laplace transform of the disk partition function (2.25). The geometrical meaning of the operator Φ is that it creates a boundary with, in general complex, boundary cosmological constant $\mu_B = x$. The vacuum expectation value $\langle \Phi(x) \rangle$ is equal to the disk partition function, the connected correlator $\langle \Phi(x)\Phi(x') \rangle_c$ gives the partition function on a cylinder, etc.

The saddle point equation (2.24) can be reformulated as a boundary condition for the current

$$J(x) = -\tfrac{2V'(x)+nV'(-x)}{4-n^2} - \partial_x \Phi(x), \quad x \in \mathbb{C}, \tag{2.29}$$

which is analytic on the complex plane cut along the interval $[-a, -b]$. Namely, the values of $J(x)$ on both sides of the cut are related by

$$J(x + i0) + J(x - i0) + nJ(-x) = 0 \quad x \in [-a, -b]. \tag{2.30}$$

The function $J(x)$ is completely determined by the boundary condition (2.30) and the first four coefficients of its Laurent expansion at infinity,

$$J(x) = \sum_{n=-\infty}^{3} J_n x^{n-1} = J_3 x^2 + J_2 x + J_1 + J_0 x^{-1} + [J(x)]_{<0}, \tag{2.31}$$

which are expressed in terms of T and κ,

$$J_0 = \kappa^2, \quad J_1 = -\tfrac{T(2-T)}{4(2-n)}, \quad J_2 = \tfrac{T-1}{2+n}, \quad J_3 = \tfrac{1}{2-n}. \tag{2.32}$$

We will split, as in [8], the function $J(x)$ in two 'chiral' pieces,

$$J(x) = J_+(x) + J_-(x), \tag{2.33}$$

satisfying simpler boundary conditions along the cut:

$$\begin{aligned} J_+(-x \pm i0) &= -e^{\pm i\pi\nu} \, J_+(x \pm i0) \\ J_-(-x \pm i0) &= -e^{\mp i\pi\nu} \, J_-(x \pm i0) \end{aligned} \qquad (b < x < a). \qquad (2.34)$$

The phase ν is defined by (1.3). The boundary condition (2.30), respectively (2.34), implies a quadratic functional identity for the current [12]

$$\left[J^2(x) + J^2(-x) + nJ(x)J(-x) \right]_{<0} \equiv (2 - n^2)[J_+(x)J_-(x)]_{<0} = 0, \qquad (2.35)$$

where $[\ \ ]_{<0}$ denotes the negative piece of Laurent expansion. Therefore $J_+(x)J_-(x)$ is an even polynomial of 4 degree whose coefficients are known functions of κ and T. In order to avoid heavy expressions we rescale x and b as

$$\hat{x} = x/a, \qquad \hat{b} = b/4a \qquad (2.36)$$

and use directly the Laurent expansion of the chiral components $J_\pm$:

$$J_\pm(x) = e^{\mp i\pi\nu/2} \left(\sum_k \hat{J}_{2k} \hat{x}^{2k-1} \pm i \sum_k \hat{J}_{2k+1} \, \hat{x}^{2k} \right). \qquad (2.37)$$

The new expansion coefficients are related to the old ones by

$$\hat{J}_{2k} = \frac{a^{2k-1}}{\sqrt{2-n}} J_{2k}, \qquad \hat{J}_{2k+1} = \frac{a^{2k}}{\sqrt{2+n}} J_{2k+1}. \qquad (2.38)$$

Then (2.35) implies the functional equation

$$J_+(x)J_-(x) = A + B\hat{x}^2 + C\hat{x}^4 \qquad (2.39)$$

where the A, B and C are expressed in terms of $\hat{J}_3, \hat{J}_2, \hat{J}_0$ and the first moment of the eigenvalue density $\hat{W}_1 \equiv \hat{J}_{-1}$:

$$A = 2\hat{W}_1 \hat{J}_3 + \hat{J}_1^2 + 2\hat{J}_0 \hat{J}_2, \quad B = 2\hat{J}_1 \hat{J}_3 + \hat{J}_2^2, \quad C = \hat{J}_3^2 \qquad (2.40)$$

2.3   A criterium for criticality

The functional equation (2.39) leads to an algebraic equation for $J(0)$:

$$J^2(0) \equiv (2 - n) \, J_+(0)J_-(0) = (2 - n) \, A \ \Rightarrow \ J(0) = \sqrt{(2 - n)A}. \qquad (2.41)$$

The value $J(0)$ is analytic function of the couplings κ^2 and T in the domain where the loop gas partition function is convergent. The critical lines are therefore given by the border of the domain of analyticity of $J(0)$. The latter is given by the condition $A = 0$, or

$$2\hat{W}_1 \hat{J}_3 + \hat{J}_1^2 + 2\hat{J}_0 \hat{J}_2 = 0. \qquad (2.42)$$

2.4   Solution along the critical line $\kappa = \kappa_{II}(T)$

The Riemann surface of $J(x)$ consists of infinitely many sheets, except for the case when ν is a rational number. The first, physical, sheet has one cut $[-a, -b]$ while all the other sheets have two cuts $[-a, -b]$ and $[b, a]$ (Fig. 2). If we find a global parametrization of the Riemann surface that resolves all branch points, then the boundary conditions (2.34) will become quasi-periodicity conditions and can be solved (in our case) in terms of theta functions.
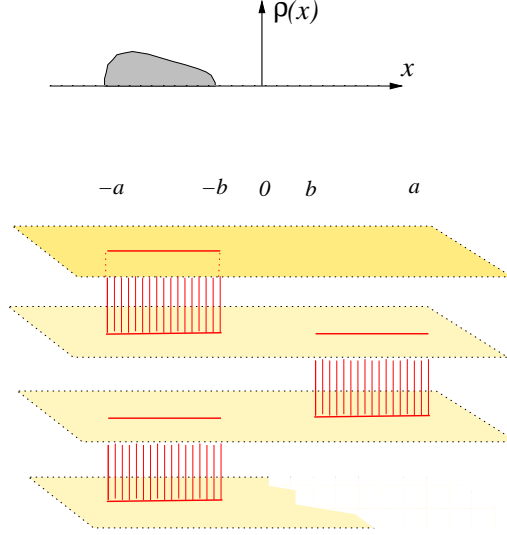

Fig. 2: The Riemann surface of $J(x)$

To illustrate the method, let us first reproduce the solution along the critical line $T = T_c(\kappa)$, found originally in [6] by applying the Wiener-Hopf method to the integral equation (2.24). Along this critical line $b = 0$ and the infinity of simple ramification points at $x = \pm b$ merge into a single ramification point of infinite order at $x = 0$. The Riemann surface is then globally parametrized by a hyperbolic map

$$\hat{x}(s) = \frac{1}{\cosh s} \tag{2.43}$$

where the points with parameters s and $-s$ must be identified. The boundary conditions (2.34) become a quasi-periodicity conditions in the complex s-plane:

$$\begin{aligned} J_+(s \pm i\pi) &= -e^{\pm i\pi\nu} \, J_+(s), \\ J_-(s \pm i\pi) &= -e^{\mp i\pi\nu} \, J_-(s). \end{aligned} \tag{2.44}$$

The unique solution of (2.44) with the asymptotics (2.37) is

$$J_\pm(s) = d_1 \frac{e^{\pm \nu s}}{\cosh s} + d_2 \frac{e^{\pm(\nu-1)s}}{\cosh^2 s}. \tag{2.45}$$

with

$$d_2 = \hat{J}_3, \quad d_1 = \hat{J}_2 - (1 - \nu)\hat{J}_3. \tag{2.46}$$

One can check that the quadratic equation (2.39) is satisfied with

$$A = 0, \quad B = d_1(2d_2 + d_1), \quad C = d_2^2.$$

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Comparing with the asymptotics (2.37) we find two more relations

$$\hat{J}_1 = \nu \hat{J}_2 - \tfrac{1-\nu^2}{2}\hat{J}_3, \qquad \hat{J}_0 = \tfrac{\nu^2}{2}\hat{J}_2 - \tfrac{1-\nu^2}{3}\hat{J}_3, \tag{2.47}$$

which determine the left branchpoint $a = a(\kappa)$ and the critical line $\kappa_c = \kappa_{II}(T)$ that describes the dense phase. The line $\kappa_c = \kappa_{II}(T)$ starts at the critical point T^* where $d_1 = 0$.

In a similar fashion we calculate the J_0-derivative of $J(z)$, which we rescale for convenience as

$$\hat{\partial}_0 J(x) \equiv a\sqrt{2-n}\ \partial_0 J(x). \tag{2.48}$$

The meromorphic functions $\hat{\partial}_0 J_\pm(x)$ satisfy the same boundary condition (2.56) but have less singular behavior at infinity:

$$\hat{\partial}_0 J_\pm(x) = \pm i e^{\mp i\pi\nu/2}\left(\tfrac{1}{\hat{x}} \mp i\tfrac{\partial_0 \hat{W}_1}{\hat{x}^2} + \ldots\right), \qquad x \to \infty. \tag{2.49}$$

The only solution of (2.56) with such asymptotics is

$$\hat{\partial}_0 J_\pm = \pm i e^{\mp\nu s}\coth s. \tag{2.50}$$

We finally obtain for the current (2.29) and its derivative

$$\begin{aligned} J &= 2d_1\tfrac{\cosh\nu s}{\cosh s} + 2d_2\tfrac{\cosh(1-\nu)s}{\cosh^2 s} \\ \hat{\partial}_0 J &= \sinh\nu s\,\coth s \end{aligned} \tag{2.51}$$

The scaling behavior for $|\hat{x}| \ll 1$ is

$$\begin{aligned} J(x) &\sim (\mu_B)^{1+\nu} - \lambda\,(\mu_B)^{1-\nu}, \\ \partial_0 J(x) &\sim (\mu_B)^\nu. \end{aligned} \tag{2.52}$$

where

$$\lambda = \tfrac{1-\nu}{2} - \tfrac{\hat{J}_2}{2\hat{J}_3} \sim T_* - T, \quad \mu_B = \hat{x}/2. \tag{2.53}$$

2.5   The general solution

A solution of the $O(n)$ model in the most general case is presented in [9,10]. However, we found easier to perform an independent computation rather than to use the results of [9,10].

For generic values of the couplings there exists a global parametrisation in terms of Jacoby elliptic functions. The uniformization map $u \to x(u)$ is defined as

$$x = \frac{b}{\mathrm{dn}(u,k)}, \qquad\qquad k' \equiv \sqrt{1-k^2} = \tfrac{b}{a} \tag{2.54}$$

(our notations are those of Gradshtein and Ryzhik [20]).

The map has the symmetries $x(u) = x(-u) = x(u+2K) = -x(u+2iK')$ and parametrizes the physical sheet by the rectangle $[0,K] \times [-2iK', 2iK']$ in the u-plane. The whole Riemann surface is parametrized by the orbifold of the u-plane with respect to the symmetry $u \to -u$.

The functions $J_\pm(u)$ must be periodic in $u \to u + 2K$

$$J_\pm(u+2K) = J_\pm(u) \tag{2.55}$$

and, by the boundary condition (2.34), quasi-periodic in $u \to u \pm 2iK'$:

$$J_+(u \pm 2iK') = -e^{\pm i\pi\nu}\,J_+(u),$$

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$$J_-(u \pm 2iK') \quad = -e^{\mp i\pi\nu} \, J_+(u).$$ (2.56)

The general solution of these equations is given in terms of the theta function $Y_\omega(u)$ defined as follows:

$$Y_\omega(u) = \frac{\Theta_1(0)}{\Theta_1(\omega K)} \frac{\Theta_1(u-\omega K)}{\Theta_1(u)} \qquad \Theta_1(u) = \theta_3\left(\frac{\pi u}{2K}\right).$$ (2.57)

The map (2.54) is a particular case of this function, $x(u) = b \, Y_1(u)$. We will need the following properties of the function $Y_\omega(u)$:

1) *Symmetries and (quasi) periodicity:*

$$Y_\omega(u) = Y_{-\omega}(-u) = Y_{\omega+2}(u) = Y_\omega(u + 2K)$$

$$Y_\omega(u \pm 2iK') = e^{\pm i\pi\omega} Y_\omega(u)$$ (2.58)

2) *Quadratic relations*

$$Y_\omega(u)Y_{-\omega}(u) = 1 + \mathrm{sd}^2(\nu K)\left(\hat{x}^2 - k'^2\right),$$ (2.59)

$$Y_{1-\omega}(u)Y_\omega(u) + Y_{-1+\omega}(u)Y_{-\omega}(u) = \tfrac{2}{k'} \, \hat{x}$$ (2.60)

$$Y_{1-\omega}(u)Y_\omega(u) - Y_{-1+\omega}(u)Y_{-\omega}(u) = \tfrac{1}{k'}\tfrac{\mathrm{sncn}}{\mathrm{dn}}(\omega K)\, \partial_u \hat{x},$$ (2.61)

3) *Particular values:*

u	x	Y_ω
0	b	1
K	a	$\mathrm{nd}(\omega K)$
$K \pm iK'$	∞	∞
$K \pm 2iK'$	$-a$	$e^{\pm i\pi\omega}\mathrm{nd}(\omega K)$
$\pm 2iK'$	$-b$	$e^{\pm i\pi\omega}$

4) *Asymptotics at $x \to \infty$:*

$$Y_\omega(x) = e^{i\pi\frac{\omega-1}{2}} \, \mathrm{sd}(\omega K) \, \left(\hat{x} - i\frac{H'(\omega K)}{H(\omega K)} + ...\right)$$ (2.62)

One can easily check that the solution of (2.56) in the case of cubic potential is given by the linear combination[3]

$$J_\pm(u) \quad = D_1 Y_{\pm(1-\nu)}(u) + D_2 Y_{\mp\nu}(u)\hat{x}(u).$$ (2.63)

Since $J_+(-u) = J_-(u)$, the function $J = J_+ + J_-$ is even function of u, hence a function on the orbifold. The coefficients D_1 and D_2 and the branch points a and $b = k'a$ are obtained by from the asymptotics at infinity given by (2.31). To simplify the expressions, we will introduce the rescaled coefficients

$$d_1 = D_1\frac{\mathrm{cn}(\nu K)}{k'}, \qquad d_2 = D_2 \, \mathrm{sd}(\nu K).$$ (2.64)

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[3]The solution for a generic polynomial potential is obtained by replacing the coefficients D_1 and D_2 with entire functions of x^2 [9].

Comparing the coefficients of the two leading of the expansion at $x \to \infty$ we find

$$d_1 = \hat{J}_2 - \frac{H'}{H}(\nu K)\hat{J}_3, \quad d_2 = \hat{J}_3. \tag{2.65}$$

Note that $(2.46) = (2.65)\mid_{k'=0}$. We need two more relations that determine the positions a and b of the branch points. Instead of expanding further around the point $x = \infty$, it is simpler to use the fact that the solution (2.63) satisfy a quadratic relation whose l.h.s. is identical to that of (2.39). This quadratic relation follows from (2.61) and (2.60):

$$\begin{aligned} J_+ J_- &= d_2^2 \hat{x}^2 (\hat{x}^2 + \mathrm{cs}^2(\nu K)) + 2d_1 d_2 \frac{\mathrm{dn}}{\mathrm{sncn}}(\nu K)\, \hat{x}^2 \\ &\quad + d_1^2(\hat{x}^2 + k'^2 \mathrm{sc}^2(\nu K)). \end{aligned} \tag{2.66}$$

Taking $x = 0$ we obtain for the coefficient A

$$A \equiv 2\hat{W}_1 \hat{J}_3 + \hat{J}_1^2 + 2\hat{J}_0 \hat{J}_2 = k'^2 \mathrm{sc}^2(\nu K) d_1^2. \tag{2.67}$$

The condition for criticality $A = 0$ is achieved either if $k' = 0$ (dense phase), or if $d_1 = 0$ (pure gravity), or if $k' = d_1 = 0$ (critical point, or dilute phase). To evaluate the boundary entropy $M \sim b(\lambda, \mu)$ we will proceed as follows. We will first calculate independently J and $\partial_\mu J$, then take the continuum limit and compare the second quantity with the derivative of the first in μ. This will give an equation for the function $b(\lambda, \mu)$.

In a similar way we evaluate the derivative $\hat{\partial} J_\pm(x)$ as the unique solution of (2.56) and (2.49). It is given by

$$\hat{\partial}_0 J_\pm = \pm i \mathrm{ds}(\nu K) \frac{Y_{\mp \nu}(u)}{\partial_u \hat{x}}. \tag{2.68}$$

where

$$\partial_u \hat{x} = i\sqrt{(\hat{x}^2 - 1)(\hat{x}^2 - k'^2)}.$$

The solution (2.68) leads to a simple formula for $\hat{\partial}_0 \partial_u \Phi$:

$$\hat{\partial}_0 \partial_u \Phi = -\mathrm{ds}(\nu K)(Y_\nu - Y_{-\nu}). \tag{2.69}$$

Finally, comparing the subleading terms of the expansions (2.49) and (2.62), we find the derivative of $W_1 = \frac{1}{\beta}\langle \mathrm{tr}\ M \rangle$,

$$-\hat{\partial}_0 \hat{W}_1 = \frac{H'(\nu K)}{H(\nu K)}, \tag{2.70}$$

whose singular part gives, up to a normalization, the susceptibility $\chi \sim -\hat{\partial}_0^2 \mathcal{F}$.

2.6 Continuum limit

The continuum limit $a \to \infty$ is achieved when both $\hat{x} = x/a$ and $\hat{b} = b/4a = k'/4$ are small. In this limit the uniformization map (2.54) degenerates to

$$x(u) = b \cosh u \tag{2.71}$$

and the theta function (2.57) is approximated by

$$Y_{\pm\omega}(u) = e^{\pm\omega u}. \tag{2.72}$$

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The two coefficients of the solution (2.63) are now given by

$$D_1 = 2\hat{b}^{1-\nu}d_1, \qquad D_2 = 2\hat{b}^{\nu}d_2 \tag{2.73}$$

and for the current $J = J_+ + J_-$ we find

$$J = 4\hat{J}_3\left(-\lambda\,\hat{b}^{1-\nu}\cosh(1-\nu)u + \hat{b}^{1+\nu}\cosh(1+\nu)u\right) \tag{2.74}$$

with the coupling λ defined earlier in (2.53). On the other hand, from (2.68) we find for the derivative $\hat{\partial}_0 J$

$$\hat{\partial}_0 J = \hat{b}^{\nu-1}\frac{\sinh\nu u}{\sinh u}. \tag{2.75}$$

Now let us compare (2.75) with the derivative of (2.74):

$$\begin{aligned}
\hat{\partial}_0 J|_x &= \hat{\partial}_0\hat{b}\left(\frac{\partial}{\partial\hat{b}} - \frac{\partial_u}{b\tanh u}\right)\cdot J \\
&= 4\hat{J}_3\,\hat{\partial}_0\hat{b}\left(-\lambda(1-\nu)\hat{b}^{-\nu} - (1+\nu)\hat{b}^{\nu}\right)\frac{\sinh\nu u}{\sinh u}.
\end{aligned} \tag{2.76}$$

The two expressions coincide under the condition

$$\hat{J}_0 - \hat{J}_0^c = -2\hat{J}_3\left(\lambda\hat{b}^{2-2\nu} + (1+\nu)\hat{b}^2\right), \tag{2.77}$$

where the T-dependent integration constant $\hat{J}_0^c$ corresponds to the value $\kappa_c = \kappa_{II}(T)$.

Eq. (2.77) is identical to the transcendental equation (1.16) with the cosmological constant μ and the boundary entropy $-M$ being normalized as

$$\mu = \frac{\hat{J}_0^c - \hat{J}_0}{2\hat{J}_3}, \qquad M = \hat{b}. \tag{2.78}$$

The susceptibility $\chi = -\partial_\mu^2\mathcal{F}$ is given, up to a constant factor, by the leading singular term in the small k' expansion of (2.70):

$$-\hat{\partial}_0\hat{W}_1 = 1 - \nu + 2\hat{b}^{2\nu} + ... \tag{2.79}$$

Since we have not yet normalized the string interaction constant $g_s \sim \beta$, we have the freedom to normalize the free energy. We do it so that $\chi = M^{2\nu}$.

Acknowledgments

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